Mixing chaotic maps and electromagnetic interference reduction

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ABSTRACT

Spread-spectrum techniques for improving Electromagnetic Compatibilities are investigated. This requires generating Constant Envelope Wideband signals. We produce them via chaotic maps that yield pseudo random time series, used afterwards to modulate sinusoidal waves in frequency. How to assess a maps’s suitability for such task is also discussed.

1 Introduction

Most digital systems need a clock to synchronize different processes. The power of a typical clock signal is concentrated on a few number of frequencies although it may produce electromagnetic interferences, which must be regulated so as to avoid malfunctioning of nearby equipment. Reducing them requires electromagnetic compatibility improvements (see, for instance, http://www.emcs.org/). Since synchronization usually resists a small deviation $\pm \triangle f$ from the frequency $f_0$ of an ideal clock, signal-processing techniques (spreading techniques) become a valid methodology for improving EMC (Callegari, Rovatti and Setti, 2003a; Lin and Chen, 1994).

The ideal signal is one that has uniform Power Density Spectrum (PDS) over $f_0 - \triangle f < f < f_0 + \triangle f$. These signals are called Constant Envelope Wideband signals. One strategy, proposed in the literature for spectrum-spreading is that of modulating in frequency (FM-
modulation) a sinusoidal clock signal via a random time series \( \{x_i\} \). It has been shown that the resulting spectrum is strongly correlated to the statistical properties of the modulating time series (Setti, Mazzini, Rovatti and Callegari, 2002; Callegari, Rovatti and Setti, 2003b; Lasota and Mackey, 1994; Beck and Schlögl, 1997).

The use of chaotic maps as pseudorandom number generators (PRNG) has attracted attention in recent years because PRNGs are widely used in many branches of science and technology. Chaotic maps have the advantage of easy implementation but the series they produce have hidden structures that make them not suitable in some applications. Recently the characterization of these hidden structures have been studied using information-theory quantifiers. Additionally, the effects of randomization techniques on the time series have been analyzed with these quantifiers (De Micco, González, Larrondo, Martín, Plastino and Rosso, 2008; De Micco, Larrondo, Plastino and Rosso, 2009). The main concepts involved are the Statistical Complexity Measure proposed by López Ruiz, Mancini and Calbet (López-Ruiz, Mancini and Calbet, 1995; Calbet and López-Ruiz, 2001), and the appropriate selection of a Symbolic Dynamics to associate a Probability Distribution Function (PDF) to the time series (Rosso, Larrondo, Martín, Plastino and Fuentes, 2007).

The main goal of this paper is to give a procedure to select chaotic time series to produce CEW signals. The associated maps used have been studied in (De Micco et al., 2008), it being shown that the Skipping procedure is better than the Discretization in the case of maps with uniform \( \rho_\infty \). Why? Because these maps already have an ideal invariant measure since they exhibit maximal normalized entropy \( (H^{hist}) = 1 \) and only a mixing parameter \( r_{mix} \) must be diminished to reach the ideal PRNG. On the contrary, Discretization is a better procedure for maps with non uniform \( \rho_\infty \) (the LOG family, for instance) because it is impossible to reach the ideal point \([1, 0]\) by Skipping.

The organization of the paper is as follows: in section 2 we present a quick overview about the generation of CEW signals and the relevant statistical properties \( (\rho_\infty \text{ and } r_{mix}) \) of mixing chaotic maps; in section 3 the estimations for \( \rho_\infty \) and \( r_{mix} \) by means of the histogram and the kneading matrix respectively are presented; in section 4 it is shown that adequate information theory quantifiers (ITQ) allows the characterization of chaotic time series; section 5 deals with the CEW signals obtained with two families of maps: the logistic family and the three-way-Bernoulli family. Each of these families contains the basic chaotic map, their iterates, and a map obtained by discretization. It is shown in this paper that the best maps, according to the proposed ITQ, produce also the best CEW signals. Finally, section 6 contains the conclusions and comments.

2 CEW signals obtained by FM-modulation of a sine-wave

As pointed out in Sec. 1, the desired power spectrum of an ideal CEW signal has a constant value for frequencies \( f \) in the range \([f_0 - \Delta f, f_0 + \Delta f]\). One way to obtain this kind of signals is by FM-modulations of a sinewave (Callegari, Rovatti and Setti, 2003a; Callegari, Rovatti and Setti, 2003b; Lin and Chen, 1994), as follows:

\[
s(t) = \sin(2\pi(f_0 + \int_{-\infty}^{t} \xi(\tau)d\tau)) ,
\]

(2.1)
\[ \xi(t) = \Delta f \sum_{k=\infty}^{\infty} x_k g(t - kT) . \]  

In Eq. (2.2) \( g(t) \) is a unit pulse of duration \( T \), \( \Delta f \) is a parameter (the frequency-detuning) determined by the synchronization tolerance, \( f_0 \) is the nominal clock rate, and \( \{x_k\} \in [-1, 1] \) is the modulating time series. Only \( T \) and the process responsible for the generation of \( \{x_k\} \) may be changed so as to optimize the power-spreading.

The power spectrum of \( s(t) \) may be evaluated by means of its autocorrelation. The derivation of the final expression is nontrivial (Callegari, Rovatti and Setti, 2001b):

\[ \phi_{ss}(\tau) = \frac{1}{2T} \left( E \left[ w^{x_0/\tau} \right] (T - \tau) g(\tau) + \int_0^T \sum_{n=1}^{\infty} g(t) g(t + \tau - nT) E_n(t, \tau) dt \right) , \]  

where \( E_n(t, \tau) = E \left[ w^{x_0(1-\tau/T) + \sum_{j=1}^{n-1} x_j + x_n(\tau/T-n)} \right] \),

and \( w = e^{2i\pi \Delta f T} \). Eqs. (2.3) and (2.4) show that the PDS depends in a complicated way on the expectation value \( E[\bullet] \) of a complex observable of the time series \( \{x_k\} \).

In order to extend these results to the chaotic case it is necessary to shortly review before the statistical properties of chaotic maps. Without loss of generality let \( \rho_0 \) be an arbitrary initial probability density and let \( \{x_k\}, k = 1, \ldots, \infty \) be the time series on \([-1, 1]\), obtained with a specific starting condition \( x_0 \) by iteration of the evolution map \( M \). Then \( x_k = M^k[x_0] \) and \( \rho_k = \mathcal{L}[\rho_0] \) where \( \mathcal{L} \) is the Perron-Frobenius Operator (PFO) associated with the map \( M \). At each iteration step a new probability density function is obtained and, under certain conditions remarked below, the series of probability density functions \( \{\rho_0, \rho_1, \cdots\} \) tends to a unique limit \( \rho_\infty \) that is independent from \( \rho_0 \). Operating conditions are (Setti et al., 2002):

1. \( \mathcal{L} \) must be positive.
2. \( \mathcal{L} \) conserves \( \|\bullet\|_1 \).
3. The PFO corresponding to the iterated map \( M^k \) is \( \mathcal{L}_k = \mathcal{L}^k \).

Properties (1) and (2) guarantee that the PFO is a mapping of a PDF into another PDF. Property (3) assures that the PFO associated with the \( k_{th} \) iterate is the \( k_{th} \) successive application of the PFO associated with the map \( M \).

\[ \rho_k = \mathcal{L}[\rho_{k-1}] = \mathcal{L}^k[\rho_0] . \]  

The advantage of studying the evolution of the PDF is that Eq. (2.5) is linear, while the original map \( M \) is nonlinear and difficult to deal with. Thus, the fact that \( \rho_\infty \) is unique implies that Eq. (2.5) has a unique fixed point with eigenvalue \( \lambda_0 = 1 \).

The maps considered in this paper are mixing maps (and then they are also ergodic). The loss of statistical dependence characterizing the time series generated by mixing maps is usually studied by considering the quantity

\[ E[\phi(x_k)\psi(x_{k+q})] = \int_X \phi(x) \psi[M^p(x)] \rho_\infty(x) dx , \]  

where
where $\phi, \psi : X \to \mathbb{R}$ are smooth functions (Baladi, 2000).

For generic mixing maps, the exact computation of $E[\phi(x_k)\psi(x_{k+q})]$ is usually an impossible task, so that one is usually restricted to determine the rate $0 < r_{mix} < 1$ of geometric convergence of Eq. (2.6) to its limiting value for $p \to \infty$:

$$r_{mix} = \max \{ |\gamma_1|, r_{ess} \} ,$$

(2.7)

where $\gamma_1$ is the isolated eigenvalue of $\mathcal{L}$ inside the unit circle, with the largest modulus, and $r_{ess}$ is given by:

$$r_{ess} = \lim_{k \to \infty} \left[ \sup_{1/k} \frac{1}{|\langle d/dxM^k \rangle|} \right]^{1/k} .$$

(2.8)

Calegari et al. (Callegari, Rovatti and Setti, 2001a) demonstrated two important theorems concerning the relation between the statical properties of the time series $x_k$ generated by a mixing map $M$, and the Power Spectrum of the FM-modulated sine wave $s(t)$. These theorems may be summarized as follows:

**Assumptions:**

Let $s(t)$ be the signal obtained by the FM-modulation process driven by a sequence of independent samples $x_k$ generated by a map with $\rho_\infty$, a pulse period $T$, and a frequency deviation $\Delta f$.

**Statement 1 (fast mixing $r_{mix} \to 0$):** If the correlations of $\{x_k\}$ are very low, the FM power spectrum $S(f)$ is given by:

$$\lim_{r_{mix} \to 0} S(f) = E_x[K_1(x, f)] + \mathrm{Re} \left\{ \frac{E^2_x[K_2(x, f)]}{1 - E_x[K_3(x, f)]} \right\} ,$$

(2.9)

where

$$E_x[f(x)] = \int f(x) \rho_\infty dx ,$$

$$K_1(x, f) = \frac{1}{2} TSinc^2(\pi T(f - \Delta f)x) ,$$

$$K_2(x, f) = e^{-i2\pi T(f - \Delta f)x} - 1 ,$$

$$K_3(x, f) = e^{-i2\pi T(f - \Delta f)x} .$$

**Statement 2 (Slow-Modulation):**

$$\lim_{T \to \infty} S(f) = \frac{1}{2\Delta f} \rho_\infty(f/\Delta f) .$$

(2.10)

Summing up, as $T$ tends to infinity and $r_{mix}$ to zero, the FM-modulated power spectra has an envelope with the same shape as $\rho_\infty$. Note that what actually counts for ascertaining the validity of (2.10) is the value of the modulation index $m = T\Delta f$, i.e., the product of $T$ and the frequency deviation $\Delta f$. Typically, it is enough to have $m$ greater than some units to find a behavior similar to that prescribed by (2.10).
3 Determination of $\rho_\infty$ and $r_{\text{mix}}$ for chaotic maps

The determination of $\rho_\infty$ and $r_{\text{mix}}$ is a difficult issue, but their estimation may be simplified through the introduction of an approximate finite-state Markov chain. Obviously, such an approach simplifies the problem and provides much less information than the original dynamical system, although its simpler structure allows for an exact computation of many properties that, in turn, constitute good estimations of those pertaining to the original system.

The procedure to obtain the finite-state Markov chain is described below (for more refined mathematical details see (Beck and Schlögl, 1997)).

- Divide the interval $[-1, 1]$ into $N_{\text{bin}}$ consecutive non-intersecting subintervals. Let $A_j$ be one of those subintervals.
- Estimate $\rho_0$: let $\{x_0\} = \{x_0^1, \ldots, x_0^G\}$ be an arbitrary set of $G$ initial conditions. If $G \gg N_{\text{bin}}$, the normalized histogram of $\{x_0\}$ is an estimation of $\rho_0$.
- Estimate $\mu_0(A_j)$: the relative frequency of the initial values in the subset $A_j \subset [-1, 1]$ is an estimation of the probability $\mu_0(A_j)$ and it is given by:
  \[
  \mu_0(A_j) = \int_{A_j} \rho_0(x) \, dx .
  \]
- Estimate $\mu_n(A_j)$: $\mu_n(A_j)$ is an estimation of the probability of finding an iterate $x_n$ in the subset $A_j$, being given by:
  \[
  \mu_n(A_j) = \int_{A_j} \rho_n(x) \, dx .
  \]
- Construct an $F \times F$ sized matrix $K$ (Drutarovsky and Galajda, 2006) with elements $K_{i,j}$ given by
  \[
  K_{i,j} = \frac{\mu(x_i \cap M^{-1}(x_j))}{\mu(x_i)}
  \]

Then the left eigenvector $\vec{\epsilon}$ of $K$ corresponding to the unitary eigenvalue is, after normalization, a discrete approximation of $\rho_\infty(x)$. The normalization is given by $\sum \epsilon_i = 1$. The ensuing PDF is then $\rho(x) = \sum \epsilon_i \chi_{X_i}(x)/\mu(x_i)$ where $\chi$ is the characteristic function of the set indicated as its subscript.

Although the knowledge of the ‘kneading matrix’ $K$ does not suffice to compute $r_{\text{mix}}$, the $K$-eigenvalue with the largest absolute value inside the unit circle represents a lower bound for it (Setti et al., 2002; Keller, 1984). It follows that the sum of the elements of any row of $K$ is always equal to 1. The meaning of the individual entries of our matrix is nice enough: the element $K_{i,j}$ represents the probability that a trajectory starting in $x_i$ falls in $x_j$ at the next time step. In the case of maps that are not Piecewise Linear, the matrix $K$ together with $\rho_\infty$ and $r_{\text{mix}}$ may be approximated in such fashion. The quality of our Markov approach depends on the partition, and a finer partition produces a better model (Froyland, 1997). Thus, a simple trial-and-error procedure may be used to construct a sequence of kneading matrices.
Figure 1: Comparison between 2D embeddings for the logistic equation and its iterates: a) logistic map $M : x_{n+1} = 4x_n(1 - x_n)$ b) $M^2$; c) $M^4$; d) $M^8$

$K$, each based on successively finer partitions, until the computed estimates reach a given prescribed accuracy. In particular $\rho_\infty$ is estimated by the normalized left eigenvector of $K$ with unit eigenvalue, and $r_{mix}$ is estimated as the eigenvalue with the second largest absolute value inside the unit circle. Summing up, a simple trial-and-error procedure may be used to construct a sequence of kneading matrices $K$, each based on successively finer partitions, until the computed estimates reach a prescribed accuracy. In particular $\rho_\infty$ is estimated by the normalized left eigenvector of $K$ with unit eigenvalue, and $r_{mix}$ is estimated as the eigenvalue with second largest absolute value inside the unit circle.

In order to diminish the value of $r_{mix}$ for a given chaotic map, Callegari et al. proposed a **Skipping** technique. This procedure consists of using an iterated map (instead of the original map). It produces a better mixing as Fig. 1 shows, where the correlation between the elements of the sequence may be recognized by the degree of coverage of an $n$-dimensional space embedding (with $n = 1, 2, \cdots$). The most uniform the coverage, the less correlated the ensuing sequence becomes and the smaller the value of $r_{mix}$. On the other hand, the **Skipping** procedure does not change the shape of $\rho_\infty$. Consequently, this technique is only useful, in the case of chaotic maps with constant $\rho_\infty$, for obtaining CEW signals.

A still another method, proposed by De Micco et al. (De Micco et al., 2008) and called the **Discretization** one is able to decrease the value of $r_{mix}$ and change $\rho_\infty$. It may be applied to maps with non-uniform $\rho_\infty$.

It consists of the following steps:

1. Normalize and discretize the sequence $x_k$ produced by the map $M$, converting it into a sequence of natural $n$-bits numbers. The operation may be expressed as follows:

$$x'_n = \text{floor} \left\{ x_n \cdot (2^n - 1) \right\}.$$  \hspace{1cm} (3.4)

Using for example 16 bits, the values produced by the Logistic equation are converted into natural numbers in the range $[0, 65555]$. This step is important for portability’s sake and also for getting a simple implementation in Integrated Circuits (ICs) or in Field Programmable Gate Arrays (FPGAs).
2. For each value discard all the bits but the least significant one. This is equivalent to classifying natural numbers into odd and even.

3. Regroup $n$-bits to get $n$-bits natural numbers.

4. Normalize them as rational numbers in $[-1, 1]$.

4 Information Theory Quantifiers (ITQ)

It is interesting to see that ITQ allow us to know how far we are from the ideal, a uniform $\rho_\infty$ plus $r_{mix} = 0$. These ITQ were proposed by several authors and have been useful to classify chaotic and stochastic systems (López-Ruiz et al., 1995; Rosso, Larrondo, Martín, Plastino and Fuentes, 2007; Lamberti, Martín, Plastino and Rosso, 2004). ITQ are functionals of the probability distribution function (PDFs) of a time-series. Let $\{x_i\}$ be the time series under analysis, with length $N_{dat}$. A suitable PDF can be assigned to it in infinitely variegated ways, a point that will be given due consideration below. In the meantime, suppose that the chosen PDF is discrete and given by $P = \{p_i; i = \cdots, N\}$. One defines then various quantities, namely,


$$S[P] = -\sum_{i=1}^{N} p_i \ln(p_i).$$

(4.1)

It is well known that the maximum $S_{max} = \ln(N)$ is obtained for $P_e = \{1/N, \cdots, 1/N\}$, that is, the uniform PDF. A “normalized” entropy $H[P]$ can also be defined in the fashion

$$H[P] = S[P]/S_{max}.$$ 

(4.2)

2. Statistical Complexity Measure. A full discussion about Statistical Complexity Measures exceeds the scope of this presentation. For a comparison amongst different complexity measures see the excellent paper by Wackerbauer et al. (Wackerbauer, Witt, Atmanspacher, Kurths and Scheingraber, 1994). In this paper we adopt the definition by López Ruiz-Mancini-Calbet seminal paper (López-Ruiz et al., 1995) with the modifications advanced in (Lamberti et al., 2004) so as to ensure that the concomitant SCM-version becomes (i) able to grasp essential details of the dynamics, (ii) an intensive quantity and, (iii) capable of discerning both among different degrees of periodicity and chaos (Rosso, Zunino, Pérez, Figliola, Larrondo, Garavaglia, Martín and Plastino, 2007). The ensuing measure, to be referred to as the intensive statistical complexity, is a functional $C[P]$ that reads

$$C[P] = Q_J[P, P_e] \cdot H[P],$$

(4.3)

where $Q_J$ is the “disequilibrium”, defined in terms of the so-called extensive Jensen-Shannon divergence (which induces a squared metric) (Lamberti et al., 2004). One has

$$Q_J[P, P_e] = Q_0 \cdot \{S[(P + P_e)/2] - S[P]/2 - S[P_e]/2\},$$

(4.4)
with $Q_0$ a normalization constant ($0 \leq Q_J \leq 1$) that reads

$$Q_0 = -2 \left\{ \left( \frac{N+1}{N} \right) \ln(N+1) - 2 \ln(2N) + \ln N \right\}^{-1}. \quad (4.5)$$

We see that the disequilibrium $Q_J$ is an intensive quantity that reflects on the system’s “architecture”, being different from zero only if there exist “privileged”, or “more likely” states among the accessible ones. $C[P]$ quantifies the presence of correlational structures as well (Martín, Plastino and Rosso, 2003; Lamberti et al., 2004). The opposite extremes of perfect order and maximal randomness possess no structure to speak of and, as a consequence, $C[P] = 0$. In between these two special instances a wide range of possible degrees of physical structure exist, degrees that should be reflected in the features of the underlying probability distribution. In the case of a PRNG the “ideal” values are $H[P] = 1$ and $C[P] = 0$.

As pointed out above, $P$ itself is not a uniquely defined object and several approaches have been employed in the literature so as to “extract” $P$ from the given time series. Just to mention some frequently used extraction procedures: a) time series histogram (Martín, 2004), b) binary symbolic-dynamics (Mischakow, Mrozek, Reiss and Szymczak, 1999), c) Fourier analysis (Powell and Percival, 1979), d) wavelet transform (Blanco, Figliola, Quiñ Quiroga, Rosso and Serrano, 1998; Rosso, Blanco, Jordanova, Kolev, Figliola, Schürmann and Başar, 2001), e) partition entropies (Ebeling and Steuer, 2001), f) permutation entropy (Bandt and Pompe, 2002; Keller and Sinn, 2005), g) discrete entropies (Amigó, Kocarev and Tomovski, 2007), etc. There is ample liberty to choose among them. In (De Micco et al., 2008) two probability distributions were proposed as relevant for testing the uniformity of $\mu(x)$ and the mixing constant: (a) a $P$ based on time series’ histograms and (b) a $P$ based on ordinal patterns (permutation ordering) that derives from using the Bandt-Pompe method (Bandt and Pompe, 2002).

For extracting $P$ via the histogram divide the interval $[-1, 1]$ into a finite number $N_{\text{bin}}$ of non-overlapping subintervals $A_i: [-1, 1] = \bigcup_{i=1}^{N_{\text{bin}}} A_i$ and $A_i \cap A_j = \emptyset \forall i \neq j$. Note that $N$ in Eq. (4.1) is equal to $N_{\text{bin}}$. Of course, in this approach the temporal order of the time-series plays no role at all. The quantifiers obtained via the ensuing PDF are called in this paper $H^{(\text{hist})}$ and $C^{(\text{hist})}$. Let us stress that for time series within a finite alphabet it is relevant to consider an optimal value of $N_{\text{bin}}$ (see i.e. (De Micco et al., 2008)).

In extracting $P$ by recourse to the Bandt-Pompe method the resulting probability distribution $P$ is based on the details of the phase-space-reconstruction procedure. Causal information is, consequently, duly incorporated into the construction-process that yields $P$. The quantifiers obtained via the ensuing PDF are called in this paper $H^{(BP)}$ and $C^{(BP)}$. A notable Bandt-Pompe result consists in getting a clear improvement in the quality of information theory-based quantifiers (Larrondo, González, Martín, Plastino and Rosso, 2005; Larrondo, Martín, González, Plastino and Rosso, 2006; Kowalski, Martín, Plastino and Rosso, 2007; Rosso, Larrondo, Martín, Plastino and Fuentes, 2007; Rosso, Zunino, Pérez, Figliola, Larrondo, Garavaglia, Martín and Plastino, 2007; Zunino, Pérez, Martín, Plastino, Garavaglia and Rosso, 2007).

The extracting procedure is as follows. For the time-series $\{x_t : t = 1, \cdots, N_{\text{dat}}\}$ and an embedding dimension $D > 1$, one looks for “ordinal patterns” of order $D$ (Bandt and Pompe,
Table 1: $r_{mix}$ as a function of the iteration-order $d$ for LOG and TWB.

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<th>$d$</th>
<th>LOG</th>
<th>TWB</th>
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<td>1</td>
<td>0.56789</td>
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<td>8</td>
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</tr>
</tbody>
</table>

Table 1: $r_{mix}$ as a function of the iteration-order $d$ for LOG and TWB.

2002; Keller and Lauffer, 2003; Keller and Sinn, 2005) generated by

$$(s) \mapsto (x_{s-(D-1)}, x_{s-(D-2)}, \ldots, x_{s-1}, x_{s}) , \quad (4.6)$$

which assign to each “time $s$” a $D$-dimensional vector of values pertaining to the times $s, s - 1, \ldots, s - (D - 1)$. Clearly, the greater the $D$-value, the more information on “the past” is incorporated into these vectors. By the “ordinal pattern” related to the time $(s)$ we mean the permutation $\pi = (r_0, r_1, \ldots, r_{D-1})$ of $(0, 1, \ldots, D - 1)$ defined by

$$x_{s-r_{D-1}} \leq x_{s-r_{D-2}} \leq \cdots \leq x_{s-r_1} \leq x_{s-r_0} . \quad (4.7)$$

In order to get a unique result we consider that $r_i < r_{i-1}$ if $x_{s-r_i} = x_{s-r_{i-1}}$. Thus, for all the $D!$ possible permutations $\pi$ of order $D$, the probability distribution $P = \{p(\pi)\}$ is defined by

$$p(\pi) = \sharp \{s|s \leq N_{dat} - D + 1; (s) \text{ has type } \pi\} \frac{1}{N_{dat} - D + 1} . \quad (4.8)$$

In the last expression the symbol $\sharp$ stands for “number”.

The advantages of the Bandt-Pompe method reside in a) its simplicity, b) the associated extremely fast calculation-process, c) its robustness in presence of observational and dynamical noise, and d) its invariance with respect to nonlinear monotonous transformations. The Bandt-Pompe’s methodology is not restricted to time series representative of low dimensional dynamical systems but can be applied to any type of time series (regular, chaotic, noisy, or reality based), with a very weak stationary assumption (for $k = D$, the probability for $x_t < x_{t+k}$ should not depend on $t$ (Bandt and Pompe, 2002)). One also assumes that enough data are available for a correct phase-space-reconstruction. Of course, the embedding dimension $D$ plays an important role in the evaluation of the appropriate probability distribution because $D$ determines the number of accessible states $D!$. Also, it conditions the minimum acceptable length $N_{dat} \gg D!$ of the time series that one needs in order to work with a reliable statistics. In relation to this last point Bandt and Pompe suggest, for practical purposes, to work with $3 \leq D \leq 7$ with a time lag $\tau = 1$. This is what we do here: we report results with $D = 6$ but we have tested the method and conclusions for $D = 3, 4$ and 5).

Summing up, the quantifiers to be compared here are: $H^{(hist)}, C^{(hist)}, H^{(BP)}, C^{(BP)}$. These quantities should tell us how good is our time-series to get a CEW signal. PRNG is compared
to the ideal condition $\rho_\infty(x) = \text{const}, r_{mix} = 0$. $H^{(\text{hist})}$ is the natural quantifier to analyze a non constant $\rho_\infty$, with value 1 for the ideal case. It does not depend on the order of appearance of a given time-series event and consequently it can not measure how mixing a time series is. Thus, to get a good representation plane we ought to demand a quantifier that changes with $r_{mix}$ and not with $\rho_\infty$. To look for such a kind of quantifier we must study $H^{(\text{hist})}, C^{(\text{hist})}, H^{(BP)}, C^{(BP)}$ as functions of $r_{mix}$. A family of iterated maps $M^d$ may be used to that end because they share the same invariant measure and $r_{mix}$ is a decreasing function of $d$. The best quantifier for $r_{mix}$ would be the one that which has maximal variation over the entire family of maps.

5 Results

We now present results for two families of chaotic maps: i) the Logistic family (LOG) and ii) the Three-Way-Bernoulli family (TWB), that have been selected, among other possibilities, because they are representative of two different classes of systems.

- LOG is given by:

$$x_{n+1} = 4x_n(1-x_n), \quad (5.1)$$

and its natural invariant density can be exactly determined:

$$\rho_{\text{inv}}(x) = \frac{1}{\pi \sqrt{x(1-x)}}. \quad (5.2)$$

LOG is paradigmatic because it is representative not only of maps with a quadratic maximum but also emerges when the Lorenz procedure is applied to many continuous attractors with basins that may be approached with the Lorenz method via a 1D-map (like the Lorenz, Rossler, Colpits ones among others). A non uniform natural invariant density is an important feature in this instance (Beck and Schlögl, 1997). The ensuing $r_{mix}$-values are displayed in Table 1. They have been obtained by approximating the $K$ matrix, as described in equation (3.3).

- TWB is given by:

$$x_{n+1} = \begin{cases} 
3x_n & \text{if } 0 \leq x_n \leq 1/3 \\
3x_n - 1 & \text{if } 1/3 < x_n \leq 2/3 \\
3x_n - 2 & \text{if } 2/3 < x_n \leq 1
\end{cases}. \quad (5.3)$$

TWB is representative of the class of piecewise-linear maps as, for example, the Four Way Tailed Shift Map, the Skew Tent Map, the Three Way Tailed Shift Map, etc. All these maps share a uniform natural invariant density (Beck and Schlögl, 1997). The mixing constant $r_{mix}$ of the whole family of maps $M^d$ is given by $r_{mix}^d = (1/3)^p$ (see table 1).

In the case of the LOG-family, $r_{mix}$ was determined by recourse to the $K$-matrix, as explained in Section 3. For the evaluation of the different quantifiers we used files with $N_{\text{dat}} = 50 \cdot 10^6$ floating point numbers. For the Bandt-Pompe approach we consider $D = 6$ while for histograms we have taken $N_{\text{bin}} = 2^{16}$. Figure 2 illustrates the behavior of all the quantifiers for the iterates of LOG (Figs. (a)) and the iterates of TWB (Figs. (b)), respectively. These figures show that
Figure 2: Information Theory quantifiers as functions of $r_{mix}$ for: (a) LOG map, (b) TWB map.

Figure 3: FM-spectrum for: (a) LOG; (b) $8_{th}$-iterate of LOG; (c) LOG discretized with 16 bits and discarding all but the LSB; (d) TWB; (e) $8_{th}$-iterate of TWB; (f) TWB discretized with 16 bits and discarding all but the LSB

$C^{(BP)}$ is useful for measuring $r_{mix}$. On the other hand, $H^{(hist)}$ and $C^{(hist)}$ depend on $\rho_{\infty}$ but do not depend on $r_{mix}$.

The results displayed in Table 1 show that, in the TWB instance, $r_{mix}$ decreases in fast fashion as the iteration process proceeds, which enables us to assert that TWB has better mixing properties than LOG. Looking at the representative points in the plane $H^{(hist)} \times C^{(BP)}$ for the $8_{th}$ iterate of each map we realize that, up to 8 significant figures, such point's coordinates are $[0, 97825084, 0.00001862]$ for LOG and $[0, 99994076, 0]$ for TWB.

The problem with the TWB, and in general, with piecewise linear maps, is that they are not straightforwardly implemented in computers (Jessa, 2002; Li, 2004) and very often exhibit parasitic stable equilibria which prevent the desired chaotic behavior of the system (Callegari, Setti and Langlois, 2003) to show up.

Figure 3 (a) to (f) compares the FM-spectrum obtained with the sequence of two chaotic maps: first column corresponds to the LOG map; and second column to the TWB map.
Figure 4: Representation plane $H^{(hist)} \times C^{(BP)}$ for both randomization processes: Discretization (full line) and Skipping (dashed line). (a) In the case of the LOG map Discretization produces the sequence that has ideal coordinates $(1, 0)$. Skipping does not improve $H^{(hist)}$ and the final coordinates are $(0.98, 0)$; (b) For the TWB map Discretization decreases the $C^{(BP)}$ to the ideal value $0$, even though it also diminishes $H^{(hist)} = 0.96$, while the Skipping technique produces a time series with the ideal value $(1, 0)$.

It is clear from our numerical results that, in the case of the LOG map, the envelope of the PDS obtained using the $8^{th}$ iterate is identical to the map’s $\rho_{mix}$ (estimated by the PDF based on the histogram), as Calegari et al.’s theorems predict. But, neither the logistic $8^{th}$ iterated PDF nor the FM PDS are flat as required. With the Discretization technique (see Fig. 3(c)) we get a flat PDS quite similar to the random ideal case. It means that for EMC the best member of the LOG family corresponds to Fig. 3(c).

In the case of the TWB family, the PDF of the original sequence is already uniform. This is reflected in the constant envelope of the FM-modulated signal PDS (see Fig. 3(d)); the Skipping technique improves on $r_{mix}$ (see Fig. 3(e)) and it does not change $\rho_{\infty}$. Moreover, as illustrated by Fig. 3(f), the Discretization procedure results in a “deteriorated” spectrum, on account of the fact that the TWB is a non-symmetric map. As a consequence, for EMC the best member of the TWB family corresponds to Fig. 3(e).

Such conclusions are confirmed by Figs. 4, that depict the representative point, in the plane $H^{(hist)} \times C^{(BP)}$, of each member of the LOG family (Fig. 4(a)) and of the TWB family (Figs. 4(b)). The horizontal axis of this representation plane, $H^{(hist)}$, is a quantifier for the histogram uniformity, and consequently a quantifier for $\rho_{\infty}$ uniformity. The vertical axis of the representation plane, $C^{(BP)}$, is a quantifier of $r_{mix}$, as was shown in section 4 (see also Fig.2). The ideal time series is the one with a representative point with $H^{(hist)} = 1$ and $C^{(BP)} = 0$ (De Micco et al., 2008; Rosso, Larrondo, Martin, Plastino and Fuentes, 2007). It is clear that the best time series of the LOG family is that obtained with Discretization while Skipping is the best choice for TWB family.
6 Conclusions

We have shown that chaotic maps can be successfully used to generate time series that produce CEW signals by FM-modulation of a sinusoidal wave. The best maps, according to the representation plane $H^{(\text{hist})} \times C^{(BP)}$ generate the best CEW signals. Starting from a basic map it is possible to obtain a family of maps by Skipping or Discretization. Information theory quantifiers allows one to select the best map of a given family. Two PDF’s must be used: i) the one of histogram-origin to evaluate $\rho_\infty$ and ii) the Bandt and Pompe PDF to evaluate $r_{\text{mix}}$. In this communication the planar representation $H^{(\text{hist})} \times C^{(BP)}$ was the one employed but we must point out that identical conclusions arise if one uses instead the representation plane $H^{(BP)} \times C^{(\text{hist})}$. Recourse to the representation plane $H^{(\text{hist})} \times C^{(BP)}$ is not necessary to explicitly evaluate the PFO or the $\mathbf{K}$ matrix that, in turn, are able to predict the quality of CEW signals implemented with a specific map.

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References


